Positroid learning seminar - Jan 9

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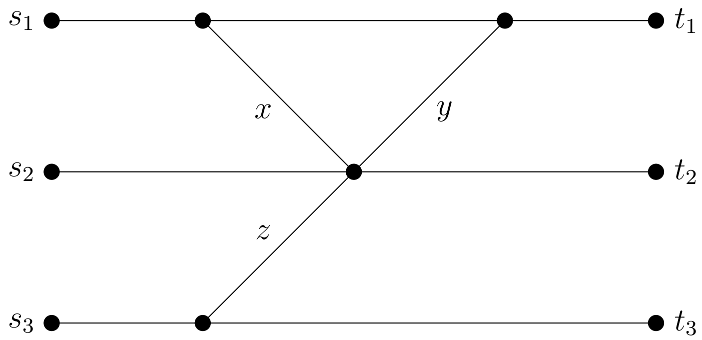
1 Motivation: Totally nonnegative matrices and acyclic networks

Our first goal is to use the Linsdröm-Gessel-Viennot Lemma to come up with a combinatorial model for the space of totally nonnegative matrices. Good references for the material in the section are the notes by Thomas Lam and this expository article by Sergei Fomin and Andrei Zelevinski.

Consider an acyclic (no oriented cycles) planar network N with n sources $s_1, \ldots, s_n$ and n sinks $t_1, \ldots t_n$. We draw such a network so that all edges are oriented left to right. Any unlabelled edge e is assumed to have weight 1, $\mathrm{wt}(e) = 1$. For an oriented path P in the network, we define $\mathrm{wt}(P) = \prod_{e \in P} \mathrm{wt}(e)$. For a collection of paths $\mathcal{C} = \{P_1, \ldots, P_m\}$ in the network, we define the weight of the collection to be $\mathrm{wt}(\mathcal{C}) = \prod_{i=1}^{m} \mathrm{wt}(P_i)$. From the network N, we build the matrix $A(N) = (a_{ij})$, where

$$a_{ij} = \sum_{P : i \to j} \mathrm{wt}(P),$$

where the sum is across all oriented paths from source s_i to sink t_j. For example, the network



gives us the matrix

$$A(N) = \begin{pmatrix} 1 + xy & x & 0 \\ y & 1 & 0 \\ yz & z & 1 \end{pmatrix}.$$

For $I, J \subset \{1, \ldots, n\}$ with $|I| = |J|$, let $A(N)_{I,J}$ denote the restriction of the matrix $A(N)$ to row set I and column set J.

Lemma 1.1 (Linsdröm-Gessel-Viennot). *Using the notation above,*

$$\det(A(N)_{I,J}) = \sum_{\mathcal{C}:I\to J} \mathrm{wt}(\mathcal{C}),$$

where the sum is over all collections of pairwise vertex disjoint paths connecting the set of sources labelled by I to the set of sinks labelled by J.

So, we see in the example above that $\det(A(N)_{13,12}) = z$. Note also that since the formal is subtraction free, if we evaluate all of the edge weight variables at positive real numbers that every minor will be nonnegative. For example, setting $x = y = z = 2$, we see that every minor of

$$\begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

must be nonnegative, which is not obvious to me just from looking at the matrix.

Definition 1.2. A matrix is *totally nonnegative* if every minor is nonnegative.

The set of totally nonnegative matrices is a monoid under multiplication. As a monoid, it is generated by matrices of the form

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the $*$'s can be any positive real number and the star in the latter two matrices can be in any position just above or just below the main diagonal.

We can find some network N such that $A(N)$ is any of these generating matrices. Further, $A(N_1)A(N_2) = A(N_1\dot{N}_2)$, where $N_1\dot{N}_2$ is the concatenation of the networks N_1 and N_2. Hence, we see every totally nonnegative matrix may be obtained as $A(N)$ for some network N. In this sense, we think of acyclic planar networks as parameterizing the space of totally nonnegative matrices.

2 The positive grassmannian and plabic graphs

Our goal is to come up with a version of this story parameterizing the totally nonnegative Grassmannian (which we won't actually define until a little later in the reading seminar, we'll just think in terms of rectangular matrices for now). To do this, we come up with a version of the Linsdröm-Gessel-Viennot Lemma for networks possibly containing cycles. Our primary reference is this paper by Kelli Talaska. Thomas Lam's notes and this paper by Alexander Postnikov are good too.

We will be concerned with $k \times n$ real matrices with $n \geq k$. Instead of wanting all minors to be nonnegative, we will only care about the maximal minors. For any $I \subseteq \{1, \ldots, n\}$ with $|I| = k$ and $k \times n$ matrix, let $\Delta_I(M)$ denote the minor obtained by restricting M to the columns labelled by I.

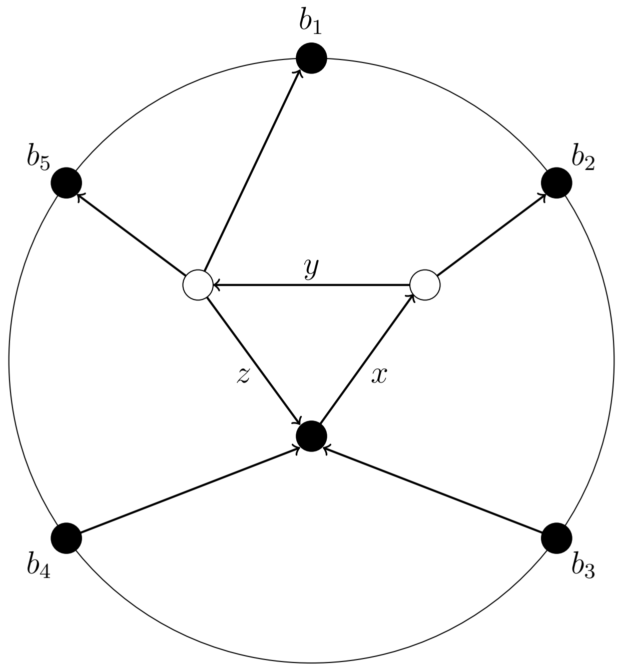
Definition 2.1. A $n \times k$ matrix M is *totally nonnegative* if $\Delta_I(M) \geq 0$ for all $I \subset \{1, \ldots, n\}$ with $|I| = k$.

We consider just a special class of networks.

Definition 2.2. A *plabic graph* (planar bicolored graph) is a planar graph embedded in a disc with boundary vertices $b_1, \ldots, b_n$ arranged clockwise along the boundary of the disc. Every boundary vertex has degree 1 and no boundary vertices are adjacent to each other. Every interior vertex is colored either black or white.

Definition 2.3. A *perfect orientation* of a plabic graph is an orientation of the edges such that each white vertex has in-degree 1 and each black vertex has out-degree 1.

The following is an example of a plabic graph with a perfect orientation. As before, all unlabelled edges are assumed to have weight 1.



Because of the oriented cycles in the graph, a path may use edges multiple times. We define the weight of a path P to be $\mathrm{wt}(P) = (-1)^{\mathrm{wind}(P)} \prod_{e \in P} \mathrm{wt}(e)$, where $\mathrm{wind}(P)$ is the topological winding index of P. We define the empty path to have weight 1. By convention, there is a path (with no edges) from any source node to itself, but there is not a path from any sink to itself (a path must start at a source). For any pair of values i, j, define $M_{i,j} = \sum_{P: i \to j} \mathrm{wt}(P)$, where the sum runs across all paths from i to j. In the example above,

$$M_{4,2} = x - x(xyz) + x(xyz)^2 - \cdots = \frac{x}{1+xyz}.$$

Note that when we evaluate the edge weights at positive numbers that $M_{i,j}$ is a positive value. Finally, we define a flow matrix $A(N) = (a_{tj})$ for our network by

$$a_{tj} = (-1)^{s(i_t,j)} M_{i_t,j},$$

where i_t is the label of the t-th source node and $s(i_t, j)$ is the number of nodes strictly between b_{i_t} and b_j counting clockwise. For the perfectly oriented plabic graph in our example,

$$A(N) = \begin{pmatrix} \frac{-xy}{1+xyz} & \frac{x}{1+xyz} & 1 & 0 & \frac{-xy}{1+xyz} \\ \frac{xy}{1+xyz} & \frac{-x}{1+xyz} & 0 & 1 & \frac{xy}{1+xyz} \end{pmatrix}.$$

Let S be the set of indices of the source nodes. For a set $I \subset \{1, \ldots, n\}$ of size $|S| = k$, define a *flow* to I to be a collection of vertex disjoint oriented paths from S to I. For any node in $S \cap I$, we use the path with no edges from that sources node to itself. Define a *conservative* flow to be a collection of vertex disjoint oriented cycles in the graph. The empty set is a conservative flow and no conservative flow can use any boundary edges. The weight of a flow or conservatives flow, $\mathrm{wt}(\mathcal{F})$, is the product of the weights of the edges used by the flow. The empty flow has weight 1. Our analog of the Lindström-Gessel-Viennot Lemma is the following.

Theorem 2.4 (Talaska). *Using the notation of above,*

$$\Delta_I(A(N)) = \sum_{\mathcal{F}:S\to I} \mathrm{wt}(\mathcal{F}) \Big/ \sum_{\mathcal{C}} \mathrm{wt}(\mathcal{C}),$$

where the first summation is across all flows from S to I and the second summation is across all conservative flows.

Using our example from above, $\Delta_{12}(A(N)) = 0$ because there are no flows from b_3, b_4 to b_1, b_2, $\Delta_{23}(A(N)) = \frac{x}{1+xyz}$. As with the Lindström-Gessel-Viennot Lemma, note that this formula is subtraction free. So, if we evaluate the edge weight variables to positive real numbers, all the maximal minors of $A(N)$ will be nonnegative. We might prove the following theorem later in the term.

Theorem 2.5 (Postinkov). *All $k \times n$ totally nonnegative matrices can be obtained using the construction above.*

3 Positroids and equivalence classes of plabic graphs

We define positroids, and give some relations for when two plabic graphs give rise to the same positroid.

Definition 3.1. A *positroid* of rank k on n elements is the set of $I \subset \{1, \dots, n\}$ with $|I| = k$ such that $\Delta_I(A(N)) \neq 0$ for some plabic graph N with a perfect orientation. Equivalently, it is the set of I such that there is a flow from the set of source nodes S to I. Equivalently, it is the set of I such that there is a perfect orientation of N with source set I (reversing the directions of all the edges in a flow to I yields a perfect orientation with source set I.

There are a whole bunch of moves and reductions we can preform on plabic graphs without altering the positroid. In fact, if one modifies edge weights in the correct way (exercise, or see the notes by Thomas Lam; these notes also contain drawing of the transformations) these transformations will alter all of the $\Delta_I(A(N))$'s by multiplication by the same scalar. Thinking of the $\Delta_I(A(N))$'s as projective coordinates (which we do...probably more on why later in the term), these transformations are equivalences on the collection $\Delta_I(A(N))$'s. If one does keep track of the edge weight properly when preforming these transformations, they will note that the numerator and denominator of every new edge weight is a subtraction free expression in the original variables. This is an example of a cluster algebra structure, which is another possible thing to explore later in the term if there is interest. The transformations are:

(i) removing or adding isolated components (not connected to any boundary vertex),

(ii) combining parallel edges into a single edge,

(iii) removing or adding degree two vertices,

(iv) removing or adding leafs which are not adjacent to boundary vertices,

(v) adding vertices to make an internal vertex of degree ≥ 3 trivalent (this a small amount of thought to see),

(vi) contracting edges between two vertices of the same color,

(vii) taking a square with white vertices on one diagonal and black on the other and flipping the colors of every vertex (also takes some thought, it's easier if you use (iv) to make everything degree three first).

Any equivalence class of transformations defines the same positroid. In fact, this map from equivalence classes of plabic graphs to positroids is bijective (maybe we'll see why later).

Using the equivalence above we can draw our plabic graphs so that every interior vertex has degree three (aside from leafs connected to boundary vertices), or we can draw our plabic graphs to be bipartite. In general, we can not do both of these things at the same time. When drawn so that every internal vertex has degree three *I think* these are what are called "on-shell diagrams" in the physics literature (my knowledge of the physics literature is somewhere between fuzzy and nonexistent).